

ON IMAGES OF MORI DREAM SPACES

SHINNOSUKE OKAWA

ABSTRACT. The purpose of this paper is to study the geometry of the images of morphisms from Mori dream spaces. Firstly we prove that a variety which admits a surjective morphism from a Mori dream space again is a MDS. Secondly we introduce a fan structure on the effective cone of a MDS and show that under a surjective morphism between MDSs the fan of the target space coincides with the restriction of the fan of the source.

We see that this fan encodes some information of the Zariski decompositions, which in turn is equivalent to the information of the GIT equivalence of VGIT of the Cox ring. Corresponding to these two interpretations, two different proofs are given to the second theorem.

Generalizations to non- \mathbb{Q} -factorial cases and to Mori dream regions are also treated.

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1. INTRODUCTION

Throughout the paper, we work over an algebraically closed base field k of characteristic $p \geq 0$.

A Mori dream space (MDS for short), whose definition and many of its basic properties was first given in the paper [HK], is a class of projective varieties with strong conditions on their line bundles. On the other hand, as proven in [HK], MDS can also be characterized as a class of varieties whose geometry of line bundles can be described in terms of the Variation of Geometric Invariant Theory quotients (VGIT for short) of its Cox ring.

The purpose of this paper is to study the geometry of the images of morphisms from MDSs. In the proofs of our results we use both of the two characterizations of MDSs mentioned above.

Our first result is the following

Theorem 1.1. *Let X, Y be normal \mathbb{Q} -factorial projective varieties, and $f : X \rightarrow Y$ be a surjective morphism. If X is a Mori dream space, then so is Y .*

We expect that Theorem 1.1 will be useful to construct new examples of MDSs.

Here we explain the structure of the proof. Theorem 1.1 will be proven by checking that a Cox ring of Y is of finite type over the base field (see Fact 5.1). We deduce it from the finite generation of a Cox ring of X . By taking the Stein factorization of f , the proof is divided into two parts: the case when f is an algebraic fiber space (§3) and the case when f is a finite morphism (§4). A finite morphism will be further decomposed into the separable part and the purely inseparable part, and treated independently (but with somewhat similar ideas). Combining them, Theorem 1.1 will be proven in §5.

Next we introduce a canonically defined fan structure on the effective cone of a MDS (see Definition-Proposition 2.8), and give it two interpretations.

One is from the view point of the Zariski decompositions of line bundles. We say that two line bundles on a MDS are strongly Mori equivalent if the negative parts of their Zariski decompositions have the same support and the positive parts define the same rational map (see Definition 2.12). With this notion, we prove that two line bundles on a MDS are strongly Mori equivalent if and only if they are contained in the relative interior of the same cone of the fan (see §2.3).

The other is in the view point of VGIT of Cox rings. To a line bundle on a MDS we associate a character of the dual torus of the Picard group, which is defined by evaluation, and we say that two line bundles on a MDS are GIT equivalent if the semi-stable loci of the corresponding characters coincide (see §7). Then we show in §7.2 that two line bundles are GIT equivalent if and only if they are strongly Mori equivalent, which in turn is equivalent to saying that they belong to the relative interior of the same cone of the fan.

Now let $f : X \rightarrow Y$ be a surjective morphism between MDSs. We denote the fan of X by $\text{Fan}(X)$ (resp. $\text{Fan}(Y)$). With the preparations above, we can show the comparison theorem of the fan of Y with that of X .

To see this, note that we can regard $\text{Pic}(Y)_{\mathbb{R}}$ as a subspace of $\text{Pic}(X)_{\mathbb{R}}$ via the natural injection

$$f^* : \text{Pic}(Y)_{\mathbb{R}} \rightarrow \text{Pic}(X)_{\mathbb{R}},$$

so that we can restrict $\text{Fan}(X)$ to the fan on $\text{Pic}(Y)_{\mathbb{R}}$ by intersecting the cones of $\text{Fan}(X)$ with $\text{Pic}(Y)_{\mathbb{R}}$. Thus we can compare $\text{Fan}(Y)$ with $\text{Fan}(X)$, and the answer turns out to be quite simple:

Theorem 1.2. *With the same assumptions as in Theorem 1.1, the fan of Y coincides with the restriction of the fan of X to the subspace $\text{Pic}(Y)_{\mathbb{R}} \subset \text{Pic}(X)_{\mathbb{R}}$: i.e.*

$$\text{Fan}(Y) = \text{Fan}(X)|_{\text{Pic}(Y)_{\mathbb{R}}}$$

holds.

See Example 9.1 for an illustration of Theorem 1.2.

We give two proofs for Theorem 1.2 corresponding to the two characterizations of the relative interiors of the cones explained above. These are treated respectively in §6 and §8. In both of the proofs, Theorem 4.1 are repeatedly used.

In the final section, we extend our results to non- \mathbb{Q} -factorial MDSs and Mori dream regions. As an example of Mori dream region, we discuss those cones spanned by finitely many klt log canonical divisors (sometimes called a Shokurov polytope). Our fan structure, in this context, reflects the information of log terminal models and log canonical models (see §10.3).

Finally we note that our results are valid in arbitrary characteristics and even when the morphisms involved are inseparable (see Example 9.2).

There are relevant preceding results to Theorem 1.1. In [B], it was shown that the projective GIT quotient of an invariant open subset of a MDS by an action of a reductive group is a MDS. When $p = 0$, it is also known (e.g. see [FG, Corollary 5.2]) that the image of a log Fano variety again is a log Fano variety. Recall that a log Fano variety is a MDS at least in characteristic zero ([BCHM, Corollary 1.3.2]).

In the paper [HK], they introduced a fan structure on the movable cone of a MDS. Our fan structure is an extension of that one. Our descriptions of the relative interiors of the cones are not explicitly treated in that paper, but the basic ideas needed for the proof implicitly appears in it.

Conventions. Unless otherwise stated, any variety is assumed to be normal and satisfy $H^0(\mathcal{O}) = k$. The word “stable” means “properly stable” in the sense of [GIT]: i.e. we assume that the stabilizer subgroup of a stable point is finite. For the notations and terminologies of MDSs and Cox rings we follow [HK], and for those of (V)GIT we follow [DH] and [GIT] (except for the definition of stability, as mentioned above).

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2. DEFINITION AND BASIC PROPERTIES OF MDS

2.1. Reminder. In this section, we briefly recall the definition and some of the basic properties of MDS which we need in this paper. For details, see [HK]. We follow the terminologies of the paper.

Definition 2.1. Let X be a normal projective variety. A small \mathbb{Q} -factorial modification (SQM, for short) of X is a small (i.e. isomorphic in codimension one) birational map $f : X \dashrightarrow Y$ to another normal \mathbb{Q} -factorial projective variety Y .

Definition 2.2. A normal projective variety X is called a Mori dream space (MDS for short) provided that the following conditions hold:

- (1) X is \mathbb{Q} -factorial, $\text{Pic}(X)$ is finitely generated and $\text{Pic}(X)_{\mathbb{Q}} \simeq N^1(X)_{\mathbb{Q}}$ holds.
- (2) $\text{Nef}(X)$ is the affine hull of finitely many semi-ample line bundles.
- (3) There is a finite collection of SQMs $f_i : X \dashrightarrow X_i$ such that each X_i satisfies (1)(2) and $\text{Mov}(X)$ is the union of the $f_i^*(\text{Nef}(X_i))$.

Remark 2.3. The author is not sure if the assumption that the morphism $\text{Pic}(X)_{\mathbb{Q}} \rightarrow N^1(X)_{\mathbb{Q}}$ is isomorphic follows from the finite generation of $\text{Pic}(X)$. We note here that it is the case at least when the base field k has uncountably many elements. In fact, finite generation of $\text{Pic}(X)$ implies that of $\text{Pic}^0(X)$. If k is uncountable, we see that $\text{Pic}_{X/k}^0$, the connected component of the identity of the Picard scheme of X , should be zero dimensional. Hence $\text{Pic}^0(X)$ is a point. By [FGA, Corollary 9.6.17], the finiteness of $\text{Pic}^{tors}(X)$, the subgroup of torsion elements of $\text{Pic}(X)$, follows from this. By [FGA, Theorem 9.6.3] and [FGA, Exercise 9.6.11], a numerically trivial line bundle should be torsion in this case. Thus we see that $\text{Pic}(X)_{\mathbb{Q}} \rightarrow N^1(X)_{\mathbb{Q}}$ is isomorphic.

When $k = \mathbb{C}$, the assumption $q = h^1(\mathcal{O}_X) = 0$ is equivalent to saying that $\text{Pic}(X)$ is finitely generated. In general, we only have the inequality $\dim \text{Pic}(X) \leq q$ (see [FGA, Corollary 9.5.13 and Remark 9.5.15].)

Let X be a normal projective variety satisfying the condition (1) of Definition 2.2. We start with recalling the definition of Mori chambers (see [HK, Definition 1.3 and 1.4]):

Definition 2.4. Let D_1 and D_2 be two \mathbb{Q} -Cartier divisors on X with finitely generated section rings. Then we say D_1 and D_2 are Mori equivalent if the rational maps

$$\varphi_{D_i} : X \dashrightarrow \text{Proj}(R_X(\mathcal{O}_X(D_i))) \quad (i = 1, 2)$$

are isomorphic: i.e. if there is an isomorphism between their target spaces which makes the obvious triangular diagram commutative.

Note that the rational map φ_{D_i} above is the same as the Iitaka fibration of D_i (in the sense of [L, Theorem 2.1.33]).

Definition 2.5. A Mori chamber of X is the closure of a Mori equivalence class in $\text{Pic}(X)_{\mathbb{R}}$ with non-empty interior.

[HK, Proposition 1.11 (2)] gives the natural decomposition of the effective cone of a MDS into Mori chambers:

Proposition 2.6. *There are finitely many contracting birational maps $g_i : X \dashrightarrow Y_i$, with Y_i a MDS, such that*

$$\text{Eff}(X) = \bigcup_i g_i^* \text{SA}(Y_i) * \text{ex}(g_i)$$

gives a decomposition of the effective cone into closed rational polyhedral subcones with disjoint interiors. Each $g_i^ \text{SA}(Y_i) * \text{ex}(g_i)$ is a Mori chamber of X .*

Above $\text{ex}(g_i)$ denotes the cone spanned by the exceptional prime divisors of g_i , and $g_i^* \text{SA}(Y_i) * \text{ex}(g_i)$ denotes the join of the cones $g_i^* \text{SA}(Y_i)$ and $\text{ex}(g_i)$. We use the notation $*$ to indicate that any element of the cone $g_i^* \text{SA}(Y_i) * \text{ex}(g_i)$ is written uniquely as the sum of the elements of the cones $g_i^* \text{SA}(Y_i)$ and $\text{ex}(g_i)$.

Here we point out some properties of the cone $\text{ex}(g_i)$.

Lemma 2.7. *For any integral divisor $E \in \text{ex}(g_i)$, $h^0(X, \mathcal{O}_X(E)) = 1$. Moreover $\text{ex}(g_i)$ is simplicial and its extremal rays are those cones spanned by an exceptional prime divisor of g_i . In particular $N_1, N_2 \in \text{ex}(g_i)$ has the same support if and only if they are contained in the relative interior of the same face of $\text{ex}(g_i)$.*

Proof. For any g_i exceptional effective divisor E , the natural map $\mathcal{O}_{Y_i} \rightarrow g_{i*} \mathcal{O}_X(E)$ is isomorphic. The first claim follows from this. The second and the third claims follow from the first one. \square

2.2. The fan of MDS. Next we introduce a fan structure on the effective cone of a MDS:

Definition-Proposition 2.8. Let X be a MDS. The set of faces of Mori chambers of X forms a fan whose support coincides with the effective cone of X . We denote it by $\text{Fan}(X)$.

Remark 2.9. The fan structure on $\text{Eff}(X)$ defined above is the extension of that on $\text{Mov}(X)$ introduced in [HK, Proposition 1.11(3)].

Proof. All we have to show is that the intersection of two cones of $\text{Fan}(X)$ is a face of each cone. Let $\sigma_1, \sigma_2 \in \text{Fan}(X)$ be two cones. We show that $\sigma_1 \cap \sigma_2$ is a face of σ_2 .

By the definition of a face, there exists classes of curves $\ell_i \in N_1(X)_{\mathbb{R}} \cong N^1(X)_{\mathbb{R}}^{\vee}$ ($i = 1, 2$) such that

$$\mathcal{C}_i \subseteq \ell_i^{\geq 0} = \{D \in N^1(X)_{\mathbb{R}} \mid D \cdot \ell_i \geq 0\}$$

and

$$\sigma_i = \mathcal{C}_i \cap \ell_i^{\perp}$$

holds for $i = 1, 2$.

Consider the following sequence of inclusions

$$\sigma_1 \cap \sigma_2 = (\mathcal{C}_1 \cap \mathcal{C}_2 \cap \ell_2^{\perp}) \cap \ell_1^{\perp} \subset \mathcal{C}_1 \cap \mathcal{C}_2 \cap \ell_2^{\perp} \subset \mathcal{C}_1 \cap \mathcal{C}_2 \subset \mathcal{C}_2.$$

From this we see that it is enough to show that $\mathcal{C}_1 \cap \mathcal{C}_2$ is a face of \mathcal{C}_2 , since we know that a face of a face again is a face (see [Fu2, page 10(4)]).

Let $g_i : X \dashrightarrow Y_i$ ($i = 1, 2$) be the contracting birational map corresponding to \mathcal{C}_i . We know that $\mathcal{C}_i = \mathcal{P}_i * \mathcal{N}_i$, where $\mathcal{P}_i = g_i^* \text{SA}(Y_i)$ and $\mathcal{N}_i = \text{ex}(g_i)$.

We divide the proof into the following claims:

Claim. (1) $\mathcal{C}_1 \cap \mathcal{C}_2 = (\mathcal{P}_1 \cap \mathcal{P}_2) * (\mathcal{N}_1 \cap \mathcal{N}_2)$.

(2) $\mathcal{P}_1 \cap \mathcal{P}_2$ is a face of \mathcal{P}_2 .

(3) $\mathcal{N}_1 \cap \mathcal{N}_2$ is a face of \mathcal{N}_2 .

(4) $\mathcal{C}_1 \cap \mathcal{C}_2$ is a face of \mathcal{C}_2 .

Proof. (4) follows from (1)(2)(3). (1) follows from the uniqueness of the Zariski decomposition of line bundles on a MDS (see Proposition 2.10). (2) is stated in [HK, Proposition 1.11(3)]. We check (3). Let $A = \sum a_i E_i$ ($a_i \geq 0$) and $B = \sum b_i E_i$ ($b_i \geq 0$) be two elements of \mathcal{N}_2 such that $A + B \in \mathcal{N}_1$. By Lemma 2.7 $h^0(X, \mathcal{O}(E)) = 1$ holds for any $E \in \mathcal{N}_1$, which means that $\text{Supp}(A + B) \subset \text{Ex}(g_1)$. Hence we see $\text{Supp}(A), \text{Supp}(B) \subset \text{Ex}(g_1)$, concluding the proof. \square

\square

2.3. Zariski decompositions and the fan. Next we give an explicit description for the Zariski decompositions (in the sense of Cutkosky-Kawamata-Moriwaki) of line bundles on a MDS, which turns out to characterize MDSs¹:

¹the author would like to thank Prof. Y.-H. Kiem for asking him if it could be the case.

Proposition 2.10. *Let X be a MDS. Consider the decomposition of $\text{Eff}(X)$ into the Mori chambers as in Proposition 2.6:*

$$\text{Eff}(X) = \bigcup_{\text{finite}} \mathcal{C}.$$

Then for each chamber \mathcal{C} there exists a small \mathbb{Q} -factorial modification $f_i : X \dashrightarrow X_i$ of X and two \mathbb{Q} -linear maps

$$P, N : \mathcal{C} \rightarrow \text{Eff}(X)$$

such that for any \mathbb{Z} -divisor $D \in \mathcal{C}$, $D \sim_{\mathbb{Q}} P(D) + N(D)$ gives the Zariski decomposition of D as a divisor on X_i ; i.e.

- $P(D) \in \text{SA}(X_i)$.
- $N(D) \geq 0$.
- The natural map

$$(1) \quad H^0(X, \mathcal{O}_X(mP(D))) \rightarrow H^0(X, \mathcal{O}_X(mD)),$$

which is defined by multiplication of a non-zero global section of the line bundle $\mathcal{O}_X(mN(D))$ is isomorphic for every sufficiently divisible positive integer m .

Zariski decomposition of D is unique up to \mathbb{Q} -linear equivalence.

Conversely a normal projective variety satisfying Definition 2.2 (1) and having a decomposition of its effective cone into finitely many chambers \mathcal{C} on which Zariski decompositions are \mathbb{Q} -linear actually is a MDS.

Proof. Let \mathcal{C} be a Mori chamber. Then we have a birational contraction $g_i : X \dashrightarrow Y_i$ to another MDS Y_i . We can replace X with one of its SQMs so that g_i becomes a morphism by (3) of Definition 2.2. Now we define the maps P, N as follows:

- $P(D) = g_i^* g_{i*} D$.
- $N(D) = D - P(D)$.

By Lemma 2.7 $h^0(X, \mathcal{O}_X(mN(D))) = 1$ holds for all sufficiently divisible positive integer m . Thus the map (1) is uniquely defined up to constant. When m is sufficiently divisible so that $mP(D)$ is a \mathbb{Z} -divisor, it is easy to see that this map has the required properties.

The uniqueness of the Zariski decomposition follows from the fact that the positive parts are movable.

The last statement can be shown by checking the finite generation of a Cox ring via exactly the same argument as in Lemma 3.2. □

Here we check that the Zariski decomposition is compatible with the pull-back of line bundles:

Lemma 2.11. *Let $f : X \rightarrow Y$ be a surjective morphism between MDSs. Let $D = P + N$ be the Zariski decomposition of an effective divisor D on Y . Then $f^*D = f^*P + f^*N$ gives the Zariski decomposition of f^*D .*

Proof. By Lemma 5.4 (2), f^*P is movable. Therefore it is enough to show that the natural mapping

$$H^0(X, \mathcal{O}_X(mf^*P)) \xrightarrow{\otimes f^*s_N^{\otimes m}} H^0(X, \mathcal{O}_X(mf^*D))$$

is an isomorphism for sufficiently divisible $m > 0$, where $s_N \in H^0(Y, \mathcal{O}_Y(N))$ is a non-zero section corresponding to the divisor N .

Replacing X and Y with their SQMs, if necessary, we may assume that P and f^*P are semi-ample. Let $c : Y \rightarrow Z$ be a birational morphism which is responsible for the Zariski decomposition of D : i.e. suppose that $D \in c^* \text{SA}(Z) * \text{ex}(c)$. Take the Stein factorization

$$X \xrightarrow{g} \tilde{Y} \xrightarrow{h} Y$$

of f , and let $\tilde{c} : \tilde{Y} \rightarrow \tilde{Z}$ be the birational morphism obtained via the Stein factorization of the morphism $c \circ h : \tilde{Y} \rightarrow Z$. We obtain the following commutative diagram:

$$\begin{array}{ccccc} X & & & & \\ \downarrow g & \searrow & & & \\ \tilde{Y} & \xrightarrow{\tilde{c}} & \tilde{Z} & & \\ \downarrow h & & \downarrow h_Z & & \\ Y & \xrightarrow{c} & Z & & \end{array}$$

Since g is an algebraic fiber space, the space of global sections of a line bundle on \tilde{Y} does not change under pull-back by g . Hence it is enough to show that the following mappings are surjective:

$$\mu : H^0(\tilde{Y}, \mathcal{O}_{\tilde{Y}}(mh^*P)) \xrightarrow{\otimes h^*s_N^{\otimes m}} H^0(\tilde{Y}, \mathcal{O}_{\tilde{Y}}(mh^*D)).$$

Take $D' \in |mh^*D|$. Then we see that $D' = \tilde{c}^*\tilde{c}_*D' + (D' - \tilde{c}^*\tilde{c}_*D')$ gives the Zariski decomposition of D' , and a section corresponding to the semi-ample divisor $\tilde{c}^*\tilde{c}_*D'$ is mapped via μ to a section corresponding to D' up to scalar multiplication, concluding the proof. \square

Now we define a stronger version of the Mori equivalence relation, which is closely related to the fan of MDSs defined above:

Definition 2.12. Let X be a MDS. Two line bundles L, M are said to be strongly Mori equivalent if they are Mori equivalent and

$$\text{Supp}(N(L)) = \text{Supp}(N(M))$$

holds, where $N(L)$ (resp. $N(M)$) is the negative part of the Zariski decomposition of L (resp. M).

Proposition 2.13. *For a MDS X , a strong Mori equivalence class coincides with the relative interior of a cone of $\text{Fan}(X)$ and vice versa.*

Proof. Let \mathcal{C} be a Mori chamber and let $\mathcal{C} = \mathcal{P} * \mathcal{N}$ be the Zariski decomposition of the chamber. By an elementary fact on convex cones, the join of a face of \mathcal{P} with a face of \mathcal{N} is a face of \mathcal{C} , and any face of \mathcal{C} is of this form. Moreover if C is a face of \mathcal{C} and $C = P * N$ is the decomposition,

$$C^{\text{relint}} = P^{\text{relint}} * N^{\text{relint}}$$

holds.

Recall also that the relative interior of a face of P is a strong Mori equivalence class. This follows from the fact that two semi-ample line bundles are Mori equivalent if and only if the set of curves contracted by the morphisms coincide.

The same thing also holds for N by Lemma 2.7.

Now Proposition 2.13 follows immediately from these facts. \square

3. FINITE GENERATION OF MULTI-SECTION RINGS ON MDS

We first prepare the notation for multi-section rings.

Definition 3.1. Let X be a normal variety with $H^0(X, \mathcal{O}_X) = k$. Let $\Gamma \subset \text{WDiv}(X)$ be a sub-semigroup of Weil divisors. The multi-section ring $R_X(\Gamma)$ associated to Γ is the Γ -graded k -algebra defined by

$$R_X(\Gamma) = \bigoplus_{D \in \Gamma} H^0(X, \mathcal{O}_X(D)).$$

Similarly for a divisor D on X , we define the section ring of D by

$$R_X(\mathcal{O}_X(D)) = \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(mD)).$$

In this section we prove the finiteness of multi-section rings on a MDS. This has first been proven in [B, Theorem 1.2] by using the finite generation theorem for invariant subrings. Our proof is based on the Zariski decompositions on MDSs, hence is more geometric.

Lemma 3.2. *Let X be a MDS. Let $\Gamma \subset \text{WDiv}(X)$ be a finitely generated group of Weil divisors. Then the multi-section ring $R_X(\Gamma)$ is of finite type over k . More generally, for any open subset $U \subset X$, $R_U(\Gamma|_U)$ is of finite type over k .*

Proof. We may assume that the natural map $\Gamma \rightarrow \text{Div}(X)_{\mathbb{Q}}$ is injective. To see this we borrow some ideas from [B].

In general we can find a splitting $\Gamma = \Gamma_0 \oplus \Gamma_1$ such that Γ_0 coincides with the kernel of $\Gamma \rightarrow \text{Div}(X)_{\mathbb{Q}}$. Then we see

$$R_X(\Gamma) \cong R_X(\Gamma_1)[\Gamma_0].$$

Note that Γ_1 maps injectively to $\text{Pic}(X)_{\mathbb{Q}}$ and that Γ_0 is a finitely generated free abelian group. Thus we may assume $\Gamma = \Gamma_1$.

Let \mathcal{C} be a Mori chamber. Set $\Gamma_{\mathcal{C}} = \Gamma \cap \mathcal{C}$. Note that it is a finitely generated semigroup. Let $g_i : X \dashrightarrow Y_i$ be the birational map corresponding to \mathcal{C} .

Recall from the proof of Proposition 2.10 that for all $D \in \Gamma_{\mathcal{C}}$

$$D = g_i^* g_{i*} D + (D - g_i^* g_{i*} D),$$

as an equality of \mathbb{Q} -divisors, gives the Zariski decomposition of D . Since Γ is finitely generated and there are only finitely many Mori chambers, there exists a positive integer $m > 0$ such that for any Mori chamber \mathcal{C} and $D \in (m\Gamma)_{\mathcal{C}}$, the positive and the negative parts of the above decomposition are both \mathbb{Z} -divisors.

We can replace Γ with $m\Gamma$, since $R_X(m\Gamma) \subset R_X(\Gamma)$ is finite.

With these preparations, we can compute $R_X(\Gamma_{\mathcal{C}})$ as follows:

$$R_X(\Gamma_{\mathcal{C}}) \cong R_Y(P(\Gamma_{\mathcal{C}}))[N(\Gamma_{\mathcal{C}})].$$

We construct an isomorphism φ from the LHS to the RHS. Choose $0 \neq s_D \in H^0(X, \mathcal{O}_X(D))$ for each $D \in N(\Gamma_{\mathcal{C}})$ such that $s_D \otimes s_{D'} = s_{D+D'}$ holds for all D, D' . Given $s \in H^0(X, \mathcal{O}_X(D))$, where $D \in \Gamma_{\mathcal{C}}$, set

$$\varphi(s) = s \otimes s_{N(D)}^{-1} \chi^{N(D)}.$$

Above $\chi^{N(D)}$ is the monomial corresponding to $N(D) \in N(\Gamma_{\mathcal{C}})$. Due to the property of the Zariski decompositions, φ is an isomorphism.

Now since $P(\Gamma_{\mathcal{C}})$ is a finitely generated semigroup of semi-ample divisors, and $N(\Gamma_{\mathcal{C}})$ is a finitely generated semigroup, we see that $R_X(\Gamma_{\mathcal{C}})$ is of finite type over k (see [HK, Lemma 2.8]).

Since there are only finitely many chambers, $R_X(\Gamma)$ itself is finitely generated over k (by the union of finite sets of generators for each $R_X(\Gamma_{\mathcal{C}})$).

Finally, the conclusion for general open subsets U follows from the case when $U = X$ (see the final part of the proof of [B, Theorem 1.2]). Note that this is the only place we need the finite generation theorem for invariant subrings. \square

4. FINITE GENERATION OF MULTI-SECTION RINGS UNDER A FINITE MORPHISM

In this section we prove that finite generation of multi-section rings is invariant under finite morphisms.

Theorem 4.1. *Let $f : X \rightarrow Y$ be a finite surjective morphism. Let $\Gamma \subset \text{WDiv}(Y)$ be a finitely generated semigroup of Weil divisors. Then the natural morphism of multi-section rings $R_Y(\Gamma) \subset R_X(f^*\Gamma)$ is finite if one of the rings is of finite type over k . Moreover, $R_Y(\Gamma)$ is of finite type over k if and only if $R_X(f^*\Gamma)$ is.*

See Definition 3.1 for the definition of multi-section rings.

4.1. Preliminary for the proof of Theorem 4.1. In the proof of Theorem 4.1, we frequently use universal torsors. We prepare some notations here.

Definition 4.2. Let $\Gamma \subset \text{WDiv}(Y)$ be a finitely generated semigroup of Weil divisors. We set

$$\mathcal{S}_Y(\Gamma) = \bigoplus_{D \in \Gamma} \mathcal{O}_Y(D),$$

and call it the universal torsor associated to Γ .

Remark 4.3.

(1) Note that

$$H^0(Y, \mathcal{S}_Y) = \bigoplus_{D \in \Gamma} H^0(Y, \mathcal{O}_Y(D)) = R_Y(\Gamma)$$

holds.

- (2) $R_Y(\Gamma)$ does not change if we replace Y with the non-singular locus of Y . When $f : X \rightarrow Y$ is a finite morphism, we may assume that X, Y are non-singular by removing suitable closed subsets of codimensions at least two. Hence we will freely assume that the varieties involved are non-singular.
- (3) Note that if we assume that Y is non-singular, then \mathcal{S}_Y is a flat \mathcal{O}_Y module.
- (4) $R_X(f^*\Gamma) = H^0(Y, f_* \mathcal{S}_X(f^*\Gamma))$, and we can calculate

$$f_* \mathcal{S}_X(f^*\Gamma) = f_* \bigoplus_{D \in \Gamma} \mathcal{O}_X(f^*D) \cong \bigoplus_{D \in \Gamma} \mathcal{O}_Y(D) \otimes_{\mathcal{O}_Y} f_* \mathcal{O}_X = \mathcal{S}_Y(\Gamma) \otimes_{\mathcal{O}_Y} f_* \mathcal{O}_X.$$

Now we go back to the proof Theorem 4.1. ‘if’ part of the second claim follows from the first claim, so we prove the first claim of Theorem 4.1 and the ‘only if’ part of its second claim.

Let

$$(2) \quad X \xrightarrow{g} \tilde{Y} \xrightarrow{h} Y$$

be the decomposition of f into the purely inseparable part g and the separable part h (i.e. \tilde{Y} is the normalization of Y in the separable closure of $k(Y)$ in $k(X)$.)

Therefore we may assume that f is either purely inseparable or separable. We treat each case separately in the following two subsections, although the ideas are basically the same.

4.2. Purely inseparable case. Assume that f is a purely inseparable morphism. We can divide the extension $k(Y) \subset k(X)$ into subextensions of degree p , so that we may assume that $\deg(f) = p$.

The key idea for this case is to describe f as a “uniform geometric quotient” by an action of a rational vector field on X :

Proposition 4.4. *Let $f : X \rightarrow Y$ be a finite surjective morphism of degree p between normal varieties. Then there exists a rational vector field on X i.e. $\delta \in \text{Der}_{k(Y)}k(X)$ such that*

$$\mathcal{O}_Y = \mathcal{O}_X^\delta := \{f \in \mathcal{O}_X \mid \delta f = 0\}$$

holds. Moreover this quotient is uniform; i.e. for a flat morphism $Z \rightarrow Y$, set $W = X \times_Y Z$ and $\delta_Z = \delta \otimes_{\mathcal{O}_Y} 1_{\mathcal{O}_Z}$. Then

$$(3) \quad \mathcal{O}_Z = \mathcal{O}_W^{\delta_Z}$$

holds.

Proof. First half is well-known (see [RS, p. 1206]). We check the uniformity (3).

Consider the following sequence of \mathcal{O}_Y modules, which is exact by definition:

$$(4) \quad 0 \rightarrow \mathcal{O}_Y \rightarrow \mathcal{O}_X \xrightarrow{\delta} k(X)$$

(since f is finite, we dropped f_*). Since Z is flat over Y , by tensoring $\otimes_{\mathcal{O}_Y} \mathcal{O}_Z$ with (4) we obtain

$$0 \rightarrow \mathcal{O}_Z \rightarrow \mathcal{O}_W \xrightarrow{\delta_Z} k(X) \otimes_{\mathcal{O}_Y} \mathcal{O}_Z,$$

concluding the proof. □

Corollary 4.5. *With the same notations as in Proposition 4.4, assume that Z and W are both normal varieties. Then*

$$H^0(W, \mathcal{O}_W)^p \subseteq H^0(Z, \mathcal{O}_Z) \subseteq H^0(W, \mathcal{O}_W)$$

holds. In particular $H^0(W, \mathcal{O}_W)$ is of finite type over k if and only if $H^0(Z, \mathcal{O}_Z)$ is, and in that case $H^0(Z, \mathcal{O}_Z) \subseteq H^0(W, \mathcal{O}_W)$ is finite.

Proof. By Proposition 4.4, we see that $W \rightarrow Z$ is a quotient by the induced vector field δ_Z on W . Therefore

$$\mathcal{O}_Z = (\mathcal{O}_W)^{\delta_Z}$$

holds. Since a derivation kills the p -th powers of functions,

$$\mathcal{O}_W^p \subseteq (\mathcal{O}_W)^{\delta_Z} = \mathcal{O}_Z$$

holds. Taking H^0 , we obtain the proof for the first line.

Next note that if R is an algebra of finite type over k , then R^p also is of finite type over k and R is a finitely generated module over R^p . ‘only if’ part of the second line follows from this. To see ‘if’ part, note that the extension $k(W) \subset k(Z)$ is finite and Z is an integral extension of W . \square

Proof of Theorem 4.1 when f is purely inseparable.

Suppose that f is purely inseparable. As mentioned before, we may assume that $\deg(f) = p$.

Note that we have a natural inclusion $\mathcal{S}_Y(\Gamma) \subset f_* \mathcal{S}_X(f^* \Gamma)$ of quasi-coherent sheaves of \mathcal{O}_Y algebras, which in turn is the product of the natural map $\mathcal{O}_Y \subset f_* \mathcal{O}_X$ with $\text{id}_{\mathcal{S}_Y(\Gamma)}$ (Remark 4.3 (4)).

By Remark 4.3 (2)(3), we may assume that X, Y are non-singular and hence $\mathcal{S}_Y(\Gamma)$ is an affine space bundle over \mathcal{O}_Y .

Therefore we can apply Corollary 4.5 for $f : X \rightarrow Y$ and the morphism $Z := \text{Spec}_{\mathcal{O}_Y} \mathcal{S}_Y(\Gamma) \rightarrow Y$. Since $H^0(Z, \mathcal{O}_Z) = R_Y(\Gamma)$ and $H^0(W, \mathcal{O}_W) = R_X(f^* \Gamma)$ (see Remark 4.3 (1) and (4)), we obtain the desired conclusions. \square

4.3. Separable case. Assume that f is separable. This case is relatively easier; by passing to a Galois closure, we can describe Y as a uniform geometric quotient by the Galois group, so that we can apply the finite generation theorem for invariant subrings.

Proof of Theorem 4.1 when f is separable. Suppose that f is separable. Let $k(W)$ be the Galois closure of $k(Y) \subset k(X)$, and let W be the normalization of X in $k(W)$. If we denote by G the Galois group of W/Y , we see that G acts on W and $Y \cong W/G$. By removing suitable closed subsets, we assume that X, Y, W are all non-singular.

Since W/X also is Galois, it is the uniform geometric quotient of the action of $\text{Gal}(W/X)$ on W . Since $\mathcal{S}_X(\Gamma)$ is locally free on X , it is a flat \mathcal{O}_X algebra, and hence we have

$$\mathcal{S}_X(\Gamma) = \mathcal{S}_W(\Gamma)^{\text{Gal}(W/X)}.$$

In particular we see that $R_X(\Gamma) = R_W(\Gamma)^{Gal(W/X)}$. Therefore $R_W(\Gamma)$ is an integral extension of $R_X(\Gamma)$.

Suppose that $R_X(\Gamma)$ is of finite type over k . By the finiteness theorem for integral closures, we see $R_W(\Gamma)$ also is of finite type over k .

Similarly, we can show that $R_Y(\Gamma) = R_W(\Gamma)^G$. Since G is a finite group, we obtain the finiteness of $R_Y(\Gamma)$.

Finally, the finiteness of $R_Y(\Gamma) \subset R_X(f^*\Gamma)$ follows from these descriptions, concluding the proof. \square

5. PROOF OF THEOREM 1.1

We prove Theorem 1.1 by the following fact, which roughly says that the finite generation of the Cox ring characterizes MDSs:

Fact 5.1. [HK, Proposition 2.9] *says that a normal projective variety satisfying Definition 2.2 (1) is a MDS if and only if a Cox ring of the variety is of finite type over k .*

First of all, we check the condition (1) of Definition 2.2 for Y ;

Lemma 5.2. *Under the same assumption as Theorem 1.1, $\text{Pic}(Y)$ is finitely generated and $\text{Pic}(Y)_{\mathbb{Q}} \simeq N^1(Y)_{\mathbb{Q}}$ holds.*

Proof. Let

$$X \xrightarrow{g} \tilde{Y} \xrightarrow{h} Y$$

be the Stein factorization of f . Let $L \in \text{Pic}(Y)$ be a numerically trivial line bundle. Then f^*L is also numerically trivial by the projection formula of intersection theory. By the assumption, there exists a positive integer m such that $f^*L^{\otimes m} \cong \mathcal{O}_X$. By taking g_* and using the projection formula, we see that $h^*L^{\otimes m} \cong \mathcal{O}_Y$. Now suppose $L \cong \mathcal{O}(D)$ for a Cartier divisor D . Since h is a finite morphism, we have $h_*h^*D \sim \deg(h)D$. Hence $m \deg(h)D \sim h_*h^*mD \sim 0$. Thus we checked the second claim.

In order to show that the finite generation of $\text{Pic}(Y)$, consider the following diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Pic}^{tors}(Y) & \longrightarrow & \text{Pic}(Y) & \longrightarrow & \text{Pic}^{free}(Y) \longrightarrow 0 \\ & & f^* \downarrow & & f^* \downarrow & & f^* \downarrow \\ 0 & \longrightarrow & \text{Pic}^{tors}(X) & \longrightarrow & \text{Pic}(X) & \longrightarrow & \text{Pic}^{free}(X) \longrightarrow 0 \end{array}$$

In the diagram above, $\text{Pic}^{tors}(X)$ is the torsion part of $\text{Pic}(X)$ and $\text{Pic}^{free}(X)$ is the quotient of $\text{Pic}(X)$ by $\text{Pic}^{tors}(X)$.

Using the similar arguments as above, we can show that the group homomorphism $f^* : \text{Pic}^{\text{free}}(Y) \rightarrow \text{Pic}^{\text{free}}(X)$ is injective. Hence we see that $\text{Pic}^{\text{free}}(Y)$ is finitely generated.

Finally we see the finiteness of the torsion part. As we checked, there exists a non-zero constant m such that for any numerically trivial line bundle L on Y $L^{\otimes m} \cong \mathcal{O}_Y$ holds. This means that $\text{Pic}^0(Y)$ is contained in the subgroup of m -division points of $\text{Pic}^0(Y)$, which is a finite set since $\text{Pic}_{Y/k}^0$ with its reduced structure is an abelian variety (see [FGA, Remark 9.5.25]). Together with [FGA, Corollary 9.6.17], the finiteness of $\text{Pic}^{\text{tors}}(Y)$ follows. \square

Remark 5.3. Using similar arguments we can directly check the following lemma, which are worth noting.

For a surjective morphism $f : X \rightarrow Y$ between normal projective varieties, $f^* : \text{Pic}(Y)_{\mathbb{R}} \rightarrow \text{Pic}(X)_{\mathbb{R}}$ is injective. We regard $\text{Pic}(Y)_{\mathbb{R}}$ as a subspace of $\text{Pic}(X)_{\mathbb{R}}$ via the mapping f^* . Then

Lemma 5.4. *With the same assumptions as in Theorem 1.1, the following equalities hold:*

- (1) $\text{Nef}(Y) = \text{Nef}(X) \cap \text{Pic}(Y)_{\mathbb{R}} = \text{SA}(X) \cap \text{Pic}(Y)_{\mathbb{R}} = \text{SA}(Y)$.
- (2) $\text{Mov}(Y) = \text{Mov}(X) \cap \text{Pic}(Y)_{\mathbb{R}} = \bigcup_i (\text{SA}(X_i) \cap \text{Pic}(Y)_{\mathbb{R}})$.
- (3) $\text{Eff}(Y) = \text{Eff}(X) \cap \text{Pic}(Y)_{\mathbb{R}}$.

Now we go back to the proof of Theorem 1.1.

Proof of Theorem 1.1. By Lemma 5.2 and Fact 5.1, it is enough to show the finiteness of a Cox ring of Y .

Set $\Gamma \subset \text{Div}(Y)$ be a finitely generated subgroup of rank $\rho(Y)$ whose image in $\text{Pic}(Y)$ is of finite index. By Lemma 3.2, we know that $R_X(f^*\Gamma)$ is of finite type over k . On the other hand

$$R_X(f^*\Gamma) \cong R_{\tilde{Y}}(h^*\Gamma)$$

holds, since g is an algebraic fiber space. Since h is finite, by Theorem 4.1 we see that $R_Y(\Gamma)$ is of finite type over k . \square

6. COMPARISON OF THE FANS -WITHOUT GIT-

In this section we prove Theorem 1.2 via direct arguments. The problem is reduced to the following

Theorem 6.1. *Let $f : X \rightarrow Y$ be a surjective morphism between MDSSs. Then two line bundles L and M on Y are strongly Mori equivalent if and only if f^*L and f^*M are strongly Mori equivalent.*

See Definition 2.12 for the notion of strong Mori equivalence.

We first check that Theorem 1.2 actually follows from this.

Proof of Theorem 1.2. Take any $\sigma \in \text{Fan}(Y)$. By Proposition 2.13 and Theorem 1.2, there exists a cone $\Sigma \in \text{Fan}(X)$ such that

$$\sigma^{\text{relint}} = \Sigma^{\text{relint}} \cap \text{Pic}(Y)_{\mathbb{R}}$$

holds. Since RHS is not empty, we can check

$$\sigma = \overline{(\Sigma^{\text{relint}} \cap \text{Pic}(Y)_{\mathbb{R}})} = \Sigma \cap \text{Pic}(Y)_{\mathbb{R}}.$$

Conversely, let $\Sigma \in \text{Fan}(X)$ be a cone which intersects with $\text{Pic}(Y)_{\mathbb{R}}$. Let Σ' be the largest face of Σ such that

$$\Sigma \cap \text{Pic}(Y)_{\mathbb{R}} = \Sigma' \cap \text{Pic}(Y)_{\mathbb{R}}$$

holds. Note that $\Sigma'^{\text{relint}} \cap \text{Pic}(Y)_{\mathbb{R}} \neq \emptyset$ holds.

Again by Proposition 2.13 and Theorem 1.2, there exists a cone $\sigma \in \text{Fan}(Y)$ such that

$$\Sigma'^{\text{relint}} \cap \text{Pic}(Y)_{\mathbb{R}} = \sigma^{\text{relint}}.$$

Taking the closures, we obtain

$$\Sigma \cap \text{Pic}(Y)_{\mathbb{R}} = \Sigma' \cap \text{Pic}(Y)_{\mathbb{R}} = \sigma.$$

□

Proof of Theorem 6.1. In view of Lemma 2.11, it is enough to show the following claim:

Claim. L, M are Mori equivalent if and only if f^*L and f^*M are Mori equivalent.

Let

$$f^*L, f^*M \in \text{Eff}(X) \cap f^* \text{Pic}(Y)_{\mathbb{R}}$$

be Mori equivalent line bundles. We prove that L, M are Mori equivalent.

First of all, take the Stein factorization of f :

$$X \xrightarrow{g} \tilde{Y} \xrightarrow{h} Y.$$

Consider the following diagram.

$$\begin{array}{ccc}
 \text{Proj } R_X(f^*M) & \xrightarrow{\sim} & \text{Proj } R_X(f^*L) \\
 \cong \downarrow & \swarrow \varphi_{f^*M} & \searrow \varphi_{f^*L} \\
 & X & \\
 \text{Proj } R_{\tilde{Y}}(h^*M) & \xrightarrow{g} & \text{Proj } R_{\tilde{Y}}(h^*L) \\
 \cong \downarrow & \swarrow \varphi_{h^*M} & \searrow \varphi_{h^*L} \\
 & \tilde{Y} &
 \end{array}$$

In the diagram above, the top horizontal arrow is an isomorphism which makes the upper triangle commutative, whose existence is guaranteed since f^*L and f^*M are Mori equivalent. Note that the two side vertical morphisms are isomorphisms, since g is an algebraic fiber space.

Now it is easy to see that the isomorphism from $\text{Proj } R_{\tilde{Y}}(h^*M)$ to $\text{Proj } R_{\tilde{Y}}(h^*L)$ which is obtained by composing the three isomorphisms in the diagram is compatible with the rational maps φ_{h^*M} and φ_{h^*L} .

Therefore we obtain the following diagram.

$$\begin{array}{ccc}
 \text{Proj } R_{\tilde{Y}}(h^*M) & \xrightarrow{\sim} & \text{Proj } R_{\tilde{Y}}(h^*L) \\
 \downarrow & \swarrow \varphi_{h^*M} & \searrow \varphi_{h^*L} \\
 & \tilde{Y} & \\
 \text{Proj } R_Y(M) & \xrightarrow{h} & \text{Proj } R_Y(L) \\
 \downarrow & \swarrow \varphi_M & \searrow \varphi_L \\
 & Y &
 \end{array}$$

In the diagram above, the top horizontal arrow is the isomorphism obtained as above, hence the top triangle is commutative. We can show that the two side vertical morphisms are finite, since h is. In fact, by Theorem 4.1 $R_{\tilde{Y}}(h^*L)$ is finite over $R_Y(L)$ (take as Γ the free abelian group generated by L . Then $R_Y(L) = R_Y(\Gamma)$ holds). The finiteness of the morphism $\text{Proj } R_{\tilde{Y}}(h^*L) \rightarrow \text{Proj } R_Y(L)$ follows from this.

Finally, there exists an isomorphism from $\text{Proj } R_Y(M)$ to $\text{Proj } R_Y(L)$ which are compatible with any other maps. In order to prove this, recall the following decomposition of the morphism h from §4:

$$\begin{array}{ccc}
 & W & \\
 & \downarrow & \searrow \\
 \tilde{Y} & \longrightarrow & S \longrightarrow Y
 \end{array}$$

In the diagram above, S is the separable closure of \tilde{Y}/Y and W is the Galois closure of S/Y .

The function field of $\text{Proj } R_Y(M)$, as a subfield of the function field of $\text{Proj } R_{\tilde{Y}}(h^*M)$, can be obtained by repeatedly taking the subfields of elements killed by vector fields which corresponds to a chain of degree p subextensions of $\tilde{Y} \rightarrow S$ (see Proposition 4.4), taking the algebraic closure in $k(W)$ (see [L, Example 2.1.12]), and taking the fields of invariants by the Galois group $G(W/Y)$ (see the arguments in §4.3). Note that these vector fields and Galois extensions depend only on the function fields of $\text{Proj } R_{\tilde{Y}}(h^*M)$, \tilde{Y} and Y . Therefore $\text{Proj } R_Y(M)$ and $\text{Proj } R_Y(L)$ should be isomorphic compatibly with the isomorphism between $R_{\tilde{Y}}(h^*M)$ and $R_{\tilde{Y}}(h^*L)$.

Conversely, take two Mori equivalent line bundles L, M on Y . We show that f^*L and f^*M are Mori equivalent. For this, we can trace back the arguments above.

Consider the following diagram:

$$\begin{array}{ccc}
 \text{Proj } R_{\tilde{Y}}(h^*M) & & \text{Proj } R_{\tilde{Y}}(h^*L) \\
 \downarrow \varphi_{h^*M} \nearrow & \tilde{Y} & \downarrow \varphi_{h^*L} \nearrow \\
 & \tilde{Y} & \\
 \downarrow h & & \downarrow \\
 \text{Proj } R_Y(M) & \xrightarrow{\cong} & \text{Proj } R_Y(L) \\
 \downarrow \varphi_M \nearrow & & \downarrow \varphi_L \nearrow
 \end{array}$$

In the diagram above, the bottom horizontal arrow ι is an isomorphism which makes the bottom triangle commutative. The existence of such an isomorphism is guaranteed by the Mori equivalence of L and M .

Note that the function field $k(\text{Proj } R_{\tilde{Y}}(h^*L))$ is the algebraic closure of $k(\text{Proj } R_Y(L))$ in $k(X)$. The same thing holds for M , so we see that the two function fields $k(\text{Proj } R_{\tilde{Y}}(h^*L))$ and $k(\text{Proj } R_{\tilde{Y}}(h^*M))$ coincide as subfields of $k(X)$.

By Theorem 4.1, three vertical morphisms are finite. Hence we see that $\text{Proj } R_{\tilde{Y}}(h^*L)$ (resp. $\text{Proj } R_{\tilde{Y}}(h^*M)$) is the normalization of $\text{Proj } R_Y(L)$ (resp. $\text{Proj } R_Y(M)$) in the same subfield of $k(\tilde{Y})$. Therefore the isomorphism ι lifts to an isomorphism between $\text{Proj } R_{\tilde{Y}}(h^*L)$ and $\text{Proj } R_{\tilde{Y}}(h^*M)$ making everything commutative. Since $g : X \rightarrow \tilde{Y}$ is an algebraic fiber space, this isomorphism guarantees the equivalence of f^*L and f^*M , concluding the proof. \square

7. MDS AND GIT REVISITED

In this section, we re-establish the relation between the variation of GIT quotients (VGIT for short) and the geometry of MDSs. To be precise, we give an explicit description for VGIT of actions of tori on affine varieties with torsion divisor class groups. After that we establish the correspondence between the fan structure defined above, and the GIT chambers and cells. The results of this section will be used later in the proof of Theorem 1.2 via GIT.

Remark 7.1. In [HK] they quote the results on VGIT from the paper [DH], in which every variety under an action by a reductive group is assumed to be proper. On the other hand for MDSs we have to deal with actions on the affine varieties defined by Cox rings, hence we need another version: namely, VGIT for affine varieties. One big difference is that for a 1-PS λ the limit $\lim_{t \rightarrow 0} \lambda(t)x$ does not exist in general. As a consequence, the wall defined by a point may not be a convex set, contrary to the proper case. In [HK] the correspondence between the GIT chambers and the Mori chambers is stated. In this paper we refine this correspondence as the equivalence of GIT equivalence with strong Mori equivalence.

7.1. VGIT of torus actions on affine varieties. The purpose of this subsection is to give an explicit description for the VGIT of actions of algebraic tori on affine varieties.

First we fix some notations.

Let G be a reductive group acting on an affine variety V . Assume for simplicity that only finitely many elements of G acts on V trivially.

We denote by $\chi(G)$ the space of characters of G , and by $\chi_{\bullet}(G)$ the space of 1-parameter subgroups of G . Note that there is a natural pairing

$$\langle \chi, \lambda \rangle = n,$$

where $\chi \in \chi(G)$, $\lambda \in \chi_{\bullet}(G)$ and $(\chi \circ \lambda)(t) = t^n$.

In the rest of this paper, for a character $\chi \in \chi(G)$ we denote by $U_\chi := V^{ss}(L_\chi)$ the semi-stable locus of V with respect to the linearization L_χ , the linearized line bundle on V whose underlying line bundle is trivial and the action of G is given by the formula

$$(g \cdot f)(x) = \chi(g)f(g^{-1}x)$$

for $g \in G$, $f \in \mathcal{O}_V$ and $x \in V$.

We denote by $q_\chi : U_\chi \rightarrow Q_\chi = U_\chi // G$ the quotient map.

Now we recall the numerical criterion for affine varieties [K, Proposition 2.5]:

Proposition 7.2. *Let G, V as above. Let $\chi \in \chi(G)$ be a character of G . Then*

- (1) $x \in V$ is L_χ semi-stable $\iff \langle \chi, \lambda \rangle \geq 0$ holds for any 1-PS $\lambda \in \chi_\bullet(G) \setminus \{0\}$ such that $\lim_{t \rightarrow 0} \lambda(t) \cdot x$ exists.
- (2) $x \in V$ is L_χ stable $\iff \langle \chi, \lambda \rangle > 0$ holds for any 1-PS $\lambda \in \chi_\bullet(G) \setminus \{0\}$ such that $\lim_{t \rightarrow 0} \lambda(t) \cdot x$ exists.

We rephrase this criteria in the case when G is an algebraic torus T . Let $V \subset \mathbb{A}$ be a T -equivariant embedding of V into an affine space. It is well-known that \mathbb{A} admits a weight decomposition $\mathbb{A} = \bigoplus_{\chi \in \chi(T)} \mathbb{A}_\chi$, where $\mathbb{A}_\chi = \{x \in \mathbb{A} | g \cdot x = \chi(g)x \ \forall g \in T\}$.

Take a point $x \in V$. According to the decomposition above, there exists a unique decomposition $x = \sum x_\chi$. Now since V is closed in \mathbb{A} , we see

$$\begin{aligned} & \lim_{t \rightarrow 0} \lambda(t) \cdot x \text{ exists in } V \\ \iff & \lim_{t \rightarrow 0} \lambda(t) \cdot x \text{ exists in } \mathbb{A} \\ \iff & \langle \chi, \lambda \rangle \geq 0 \ \forall \chi \text{ such that } x_\chi \neq 0. \end{aligned}$$

Let $\text{st}(x) = \{\chi \in \chi(T) | x_\chi \neq 0\} \subset \chi(T)$ be the state set of the point $x \in V$. Note that there are only finitely many possibility for the set $\text{st}(x)$, since it is a subset of the finite set $\text{st}(\mathbb{A}) = \{\chi \in \chi(T) | \mathbb{A}_\chi \neq 0\}$.

Denote by $\mathcal{D}_x \subset \chi(T)_{\mathbb{R}}$ the cone spanned by $\text{st}(x)$. Then

Proposition 7.3. *For a character $\chi \in \chi(T)$,*

- (1) $x \in V$ is semi-stable with respect to L_χ if and only if $\chi \in \mathcal{D}_x$ holds.
- (2) $x \in V$ is stable with respect to L_χ if and only if $\chi \in \mathcal{D}_x^\circ$, where \mathcal{D}_x° denotes the interior of the set \mathcal{D}_x (possibly empty).

Proof. This is almost tautological. From Proposition 7.2 and the argument above, $x \in V$ is semi-stable with respect to χ if and only if $\langle \chi, \lambda \rangle \geq 0$ holds for all 1-PS λ which is semi-positive definite on the cone \mathcal{D}_x : i.e. the set of such characters χ is the double dual cone of

the cone \mathcal{D}_x . Since \mathcal{D}_x is rational polyhedral, the double dual coincides with itself by [Fu2, (1) on page 9].

Stable case can be checked similarly. \square

Set $C = C^T(V) = \bigcup_{x \in V} \mathcal{D}_x$. We define the following notions according to [DH].

Definition 7.4. A wall defined by $x \in V$ is the set $\partial\mathcal{D}_x$. A GIT chamber is a connected component of the set $C \setminus \bigcup_{x \in V} \partial\mathcal{D}_x$.

Two characters χ, χ' are said to be wall equivalent if $V^{sss}(\chi) = V^{sss}(\chi')$ holds. A connected component of a wall equivalence class, which is not a chamber, is called a (GIT) cell.

Definition 7.5. Two characters χ, χ' are said to be GIT equivalent if $V^{ss}(\chi) = V^{ss}(\chi')$ holds.

Via similar arguments as in [DH, Theorem 3.3.2] and [DH, Lemma 3.3.10], we can check the following

Lemma 7.6. (1) A GIT chamber is a GIT equivalence class.

(2) For any GIT chamber \mathcal{C} ,

$$\mathcal{C} = \bigcap_{x \in V^s(\mathcal{C})} \mathcal{D}_x^\circ$$

holds.

(3) A cell is contained in a GIT equivalence class.

7.2. Strong Mori equivalence = GIT equivalence. Let X be a MDS, and fix a Cox ring $R = R_X(\Gamma)$, where Γ is a finitely generated group of Weil divisors as usual.

The purpose of this subsection is to show that the following three kinds of sets are the same:

- the relative interior of a cone of $\text{Fan}(X)$.
- a strong Mori equivalence class.
- a GIT equivalence class.

In the paper [HK], they proved this fact only for the cones of maximal dimension. We need the refined version as above for the two proofs of Theorem 1.2.

We first recall some basic facts about the relationship between MDS and VGIT (see [HK] for detail).

Reminder 7.7. Set $V_X = \text{Spec}(R_X(\Gamma))$. Recall that the torus $T_X := \text{Hom}_{gp}(\Gamma, k^*)$ acts naturally on V_X as follows: for any element $g \in T$, a divisor $D \in \Gamma$ and $f \in H^0(X, \mathcal{O}_X(D))$, set

$$g \cdot f = g(D)f.$$

As stated in [HK, Theorem 2.3], we have a natural isomorphism

$$\psi : \chi(T_X)_{\mathbb{R}} \rightarrow \Gamma_{\mathbb{R}} \cong \text{Pic}(X)_{\mathbb{R}}$$

in such a way that $D \in \Gamma$ corresponds to the character $\text{ev}_D : g \mapsto g(D)$.

For any $D \in \Gamma$, it is easy to see that $Q_{\text{ev}_D} = \text{Proj}_X(\mathcal{O}_X(D))$. In particular, if we take an ample $A \in \Gamma$, $q_{\text{ev}_A} = \text{id}_X$ holds.

By the universal property of categorical quotients, we obtain the rational map

$$X = Q_{\text{ev}_A} \dashrightarrow Q_{\text{ev}_D},$$

and it is easy to see that this coincides with the rational map $\varphi_{\mathcal{O}_X(D)} : X \dashrightarrow \text{Proj}(R_X(\mathcal{O}_X(D)))$.

Now we can state our main observation:

Proposition 7.8. *Two line bundles L, M on X are strongly Mori equivalent if and only if $U_{\text{ev}_L} = U_{\text{ev}_M}$ i.e. ev_L and ev_M are GIT equivalent. It is also equivalent to saying that the two line bundles are contained in the relative interior of the same cone of $\text{Fan}(X)$.*

Proof. The last line follows from Proposition 2.13. For the first line, the arguments in [HK, Proof of Theorem 2.3] literally works: in the proof they only proved that the relative interiors of the Mori chambers are identified (via ψ) with the GIT chambers, but the arguments can be applied more generally for arbitrary strong Mori equivalence classes.

We only sketch the proof (see [HK, Proof of Theorem 2.3] for detail).

Fix a character χ which corresponds to an ample line bundle on X . For an arbitrary character $y \in C^T(V) \cap \chi(T)$, let $\psi(y) = P + N$ be the Zariski decomposition of the corresponding \mathbb{Q} -line bundle.

Then we can show that

$$U_{\chi} \setminus U_y = q_{\chi}^{-1}(\text{Supp}(N))$$

holds in codimension one.

This follows from the following equality

$$H^0(X, \mathcal{O}_X(m\psi(y))) = H^0(U_{\chi}, L_y^{\otimes m})^T,$$

which is the same equation as (2.3.2) in [HK, Proof of Theorem 2.3], as pointed out there. From this we can immediately conclude that GIT equivalence implies the strong Mori equivalence. Conversely if we assume the strong Mori equivalence of $\psi(y)$ and $\psi(z)$ for two characters y and z , then we see that $Q_y = Q_z$ and that U_y and U_z coincide in codimension one. The rest of the arguments is precisely the same as in [HK, Proof of Theorem 2.3]. \square

Corollary 7.9. *Any GIT equivalence class is contained in a GIT cell. Combined with (3) of Lemma 7.6, this means that a GIT equivalence class and a cell are the same thing in this case.*

Proof. Take $\sigma \in \text{Fan}(X)$. If σ^{relint} is not contained in a cell, the stable loci are not constant on it: i.e. there exists a point $x \in V_X$ such that $\mathcal{D}_x^\circ \cap \sigma^{\text{relint}} \neq \emptyset$ but $\sigma^{\text{relint}} \not\subset \mathcal{D}_x^\circ$. Since \mathcal{D}_x and σ are rational polyhedral cones, this means $\sigma^{\text{relint}} \not\subset \mathcal{D}_x$, contradicting the fact that σ^{relint} is a GIT equivalence class. \square

Question 7.10. Our proof of Corollary 7.9 relies on the explicit description of GIT equivalence classes, hence is valid only for VGIT of Cox rings. It would be interesting to generalize the corollary to arbitrary VGIT of torus actions on affine varieties.

8. COMPARISON OF THE FANS -VIA GIT-

In this section we prove Theorem 1.2 via the GIT interpretation of the relative interiors of the cones (see Proposition 7.8).

To carry out the proof, we prepare some notations. Let $f : X \rightarrow Y$ be a surjective morphism between MDSs. Fix a subgroup $\Gamma \subset \text{Div}(Y)$ of rank $\rho(Y)$ which maps injectively to $\text{Pic}(Y)_{\mathbb{Q}}$. Fix also a subgroup $\Gamma_X \subset \text{Div}(X)$ of rank $\rho(X)$ mapping injectively to $\text{Pic}(X)_{\mathbb{Q}}$, and $f^*\Gamma \subset \Gamma_X$.

For such a pair of (Γ, Γ_X) , the natural ring homomorphism $f^* : R_Y(\Gamma) \rightarrow R_X(\Gamma_X)$ induces the morphism of affine varieties

$$V_f : V_X = \text{Spec } R_X(\Gamma_X) \rightarrow V_Y = \text{Spec } R_Y(\Gamma).$$

Set $T_X = \text{Hom}(\Gamma_X, k^*)$ and $T_Y = \text{Hom}(\Gamma, k^*)$. Via $f^*\Gamma \subset \Gamma_X$, we obtain the surjective morphism of algebraic tori $T_f : T_X \rightarrow T_Y$. T_X (resp. T_Y) acts on V_X (resp. V_Y) via the grading, and V_f is an equivariant morphism with respect to these actions of tori and T_f .

The following is the main ingredient of the proof of Theorem 1.2:

Proposition 8.1. *Let $f : X \rightarrow Y$ be a surjective morphism between MDSs. Then if we choose an appropriate pair (Γ, Γ_X) as above, the following holds:*

Let $V_f : V_X \rightarrow V_Y$ be the associated morphism, and $\text{ev}_L \in \chi(T_Y)$ be the character corresponding to a line bundle L on Y . Then

$$(5) \quad V_f^{-1}(V_Y^{ss}(\text{ev}_L)) = V_X^{ss}(\text{ev}_{f^*L})$$

holds.

Remark 8.2.

$$V_f^{-1}(V_Y^{sss}(\text{ev}_L)) = V_X^{sss}(\text{ev}_{f^*L})$$

is not correct in general.

The conclusion of Proposition 8.1 does not hold for an arbitrary equivariant morphism between affine varieties. For example, consider the morphism

$$\varphi : \mathbb{A}^n \rightarrow \mathbb{A}^1; (x_1, \dots, x_n) \mapsto x_1,$$

action of \mathbb{G}_m on both sides with weights one, and the character χ of weight one. Then

$$(\mathbb{A}^n)^{ss}(\chi) = \mathbb{A}^n \setminus \{0\} \supsetneq (\mathbb{A}^1 \setminus \{0\}) \times \mathbb{A}^{n-1} = \varphi^{-1}((\mathbb{A}^1)^{ss}(\chi)).$$

The following is the GIT counterpart of Theorem 6.1

Corollary 8.3. *With the same assumptions as above, let L, M be line bundles on Y . Then $V_Y^{ss}(\text{ev}_L) = V_Y^{ss}(\text{ev}_M)$ holds if and only if $V_X^{ss}(\text{ev}_{f^*L}) = V_X^{ss}(\text{ev}_{f^*M})$ holds.*

Proof. This follows from Proposition 8.1 and the surjectivity of V_f . \square

Now it is clear that Theorem 1.2 immediately follows from Corollary 8.3, in view of Proposition 7.8.

In the rest of this section, we prove Proposition 8.1. The following lemma is the key to the proof:

Lemma 8.4. *Let G be a reductive group. Let $\pi : Z \rightarrow W$ be a finite morphism between affine varieties such that G acts on Z and W equivariantly. Let \mathcal{L} be a linearization on W . Then*

$$\pi^{-1}(W^{ss}(\mathcal{L})) = Z^{ss}(\pi^*\mathcal{L})$$

holds.

Proof. See the proof for [GIT, Theorem 1.19] (and [GIT, Appendix to Chapter 1, §C] for positive characteristic cases). \square

Proof for Proposition 8.1. We need some preparation. Take the Stein factorization

$$X \xrightarrow{g} \tilde{Y} \xrightarrow{h} Y$$

of f . Fix a subgroup $\Gamma_{\tilde{Y}} \subset \text{Div}(\tilde{Y})$ of rank $\rho(\tilde{Y})$ which maps injectively to $\text{Pic}(\tilde{Y})_{\mathbb{Q}}$ and containing $h^*\Gamma$. Similarly take $\Gamma_X \subset \text{Div}(X)$ of rank $\rho(X)$ which maps injectively to $\text{Pic}(X)_{\mathbb{Q}}$ and containing $g^*\Gamma_{\tilde{Y}}$. Define groups Q and R by the following exact sequences:

$$\begin{aligned} 0 \rightarrow \Gamma &\xrightarrow{h^*} \Gamma_{\tilde{Y}} \rightarrow Q \rightarrow 0 \\ 0 \rightarrow \Gamma_{\tilde{Y}} &\xrightarrow{g^*} \Gamma_X \rightarrow R \rightarrow 0. \end{aligned}$$

Taking the duals of these sequences, we obtain the following exact sequences of algebraic tori:

$$\begin{aligned} 0 \rightarrow T_Q \rightarrow T_{\tilde{Y}} \rightarrow T_Y \rightarrow 0 \\ 0 \rightarrow T_R \rightarrow T_X \rightarrow T_{\tilde{Y}} \rightarrow 0. \end{aligned}$$

In particular, we obtain the following sequence of surjective group homomorphisms

$$(6) \quad T_X \xrightarrow{T_g} T_{\tilde{Y}} \xrightarrow{T_h} T_Y.$$

Next consider the following diagram.

$$\begin{array}{ccc} V_X & & \\ \pi_R \downarrow /T_R & & \\ J \xrightarrow{\cong} V_{\tilde{Y}} & & \\ \pi_Q \downarrow /T_Q & & \\ I \xrightarrow[\text{finite}]{} V_Y & & \end{array}$$

In the diagram above $I = \text{Spec}(R_{\tilde{Y}}(h^*\Gamma))$ and $J = \text{Spec}(R_X(g^*\Gamma_{\tilde{Y}}))$, and π_R and π_Q are natural projections. It is easy to see that they are categorical quotients by T_R and T_Q , respectively.

Note that T_X acts on V_X , $T_{\tilde{Y}}$ on J and $V_{\tilde{Y}}$, and T_Y on I and V_Y . Moreover these actions are compatible with respect to the group homomorphisms (6). Since these homomorphisms are surjective, semi-stability of points on V_Y with respect to the action of T_Y is equivalent to the semi-stability with respect to the action of T_X (similar for the points on I , $V_{\tilde{Y}}$, and J respectively).

Now we go back to the proof of Proposition 8.1.

By Lemma 8.4, we see that

$$F^{-1}(V_Y^{ss}(ev_L)) = I^{ss}(ev_{h^*L})$$

holds, since $I \rightarrow V_Y$ is finite (Theorem 4.1).

Next we prove the following

Claim.

$$\pi_Q^{-1}(I^{ss}(ev_{h^*L})) = V_{\tilde{Y}}^{ss}(ev_{h^*L}).$$

Proof of the claim. The inclusion \subseteq is the direct consequence of the definition of semi-stability, since π_Q is affine. Conversely, suppose that $x \in V_{\tilde{Y}}^{ss}(ev_{h^*L})$. Then there exists a non-zero section $s \in R_{\tilde{Y}}(\Gamma_{\tilde{Y}})$ which is semi-invariant with respect to the character ev_{h^*L} and $s(x) \neq 0$ holds. Note that semi-invariance of s with respect to the character

ev_{h^*L} and the action of $T_{\tilde{Y}}$ implies that s is the global section of some positive multiple of h^*L , hence $s \in R_{\tilde{Y}}(h^*\Gamma)$. Thus we obtain the other inclusion. \square

Since $J \rightarrow V_{\tilde{Y}}$ is isomorphic, there is nothing to argue.

By arguing as in Claim, we can show that

$$\pi_R^{-1}(J^{ss}(ev_{f^*L})) = V_X^{ss}(ev_{f^*L})$$

holds. Summing up, we obtain the desired equality. \square

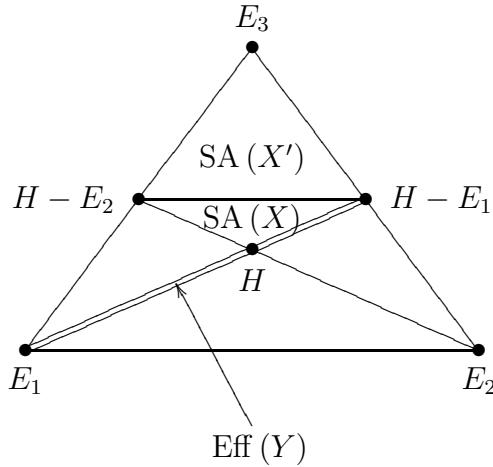
9. EXAMPLES

Example 9.1. We borrow from [AW, Example 5.5].

Let X be the blow-up of \mathbb{P}^3 in two distinct points, say p_1 and p_2 . X is toric, hence is a MDS. Let E_1, E_2 be the exceptional divisors corresponding to p_1, p_2 respectively, and let ℓ be the line passing through the points p_1, p_2 . Let E_3 be the class of the strict transformation of a plane containing ℓ . We can show that X has a flopping contraction which contracts the strict transformation of the line ℓ . Let X' be the flop. Using the toric description, we see that this is an Atiyah flop.

The effective cone of X is spanned by the divisors E_i , and the movable cone is the union of the semi-ample cones of X and X' . $\text{SA}(X)$ is spanned by three divisors $H, H - E_1$, and $H - E_2$. $\text{SA}(X')$ is spanned by $H - E_1, H - E_2$, and E_3 .

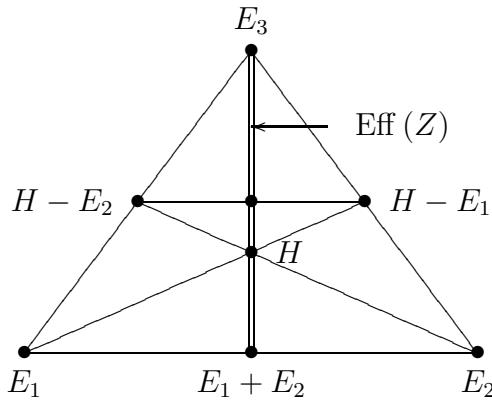
A slice of $\text{Eff}(X)$, together with its Mori chamber decomposition is described in the following figure:



Let Y be the blow-up of \mathbb{P}^3 in p_1 . Then the effective cone of Y , together with its decomposition into Mori chambers sits in $\text{Eff}(X)$ as indicated in the figure above ($\text{Eff}(Y)$ is denoted by the double line).

As indicated in the figure above, $\text{Eff}(Y)$ is mapped onto the cone spanned by E_1 and $H - E_1$. The cone spanned by H and $H - E_1$ is the semi-ample cone of Y , and that spanned by H and E_1 corresponds to the Mori chamber of Y whose interior points correspond to the line bundles defining the birational contraction to \mathbb{P}^3 .

Now take a coordinate on \mathbb{P}^3 such that $p_1 = (0 : 0 : 0 : 1)$ and $p_2 = (0 : 0 : 1 : 0)$. Consider the action of \mathbb{Z}_2 on \mathbb{P}^3 defined by $(x : y : z : w) \mapsto (x : y : w : z)$. This action lifts to X , and let $X \rightarrow Z$ be the quotient morphism. The effective cone of Z together with its Mori chamber decomposition sits in that of X as follows ($\text{Eff}(Z)$ is denoted by the double line):



As indicated in the diagram above, we can see that Z has two Mori chambers other than the semi-ample cone (recall that $\text{SA}(Z)$ coincides with the restriction of $\text{SA}(X)$ to $\text{Pic}(Z)_{\mathbb{R}}$. See Lemma 5.4(1)).

Let Z' be the quotient of X' by the involution induced from that on X . Again by Lemma 5.4(1), we can check that the Mori chamber of Z obtained by restricting $\text{SA}(X')$ is the semi-ample cone of Z' . The morphism defined by the ray separating $\text{SA}(Z)$ and $\text{SA}(Z')$ is the flipping contraction of Z which contracts the image of ℓ under the quotient morphism $X \rightarrow Z$, and Z' is the flip.

This example shows that a Mori chamber of the target space Z is not necessarily a face of a Mori chamber of the source X .

Example 9.2. This example is well-known to experts, but we give a detailed explanation for the sake of completeness. The author learned this example from Doctor Tadakazu Sawada.

Suppose that $p > 0$. Let $\mathbb{A}_{x,y}^2 \subset \mathbb{P}^2$ be an standard embedding of affine 2-plane with coordinate functions x and y . Take $f = f(x, y) \in$

$k[x, y]$. Consider the rational vector field defined by

$$\delta = \frac{\partial f}{\partial y} \frac{\partial}{\partial x} - \frac{\partial f}{\partial x} \frac{\partial}{\partial y}.$$

According to [RS], we obtain the quotient of \mathbb{P}^2 by δ . That is, we obtain a purely inseparable finite morphism $\pi : \mathbb{P}^2 \rightarrow Y$ of degree p to a normal projective variety Y such that $\mathcal{O}_Y = \mathcal{O}_{\mathbb{P}^2}^\delta$. It is easy to see that $k[x, y]^\delta = k[x^p, y^p, f(x, y)] \simeq k[X, Y, Z]/(Z^p - f(X, Y))$ holds.

Set $f(x, y) = x^p y + x y^p$. By the Fedder's criterion for F -purity ([F, Proposition 1.7]), we can check that the singularity $(0 \in k[x, y]^\delta) \cong k[[X, Y, Z]]/(Z^p - X^p Y - X Y^p)$ is not F -pure. Therefore Y is not globally F -regular, despite \mathbb{P}^2 is.

On the other hand we can show that Y is a MDS. Firstly Picard number of Y is one since $\pi^* : \text{Pic}(Y)_{\mathbb{R}} \rightarrow \text{Pic}(\mathbb{P}^2)_{\mathbb{R}}$ is injective. We can also check that Y is \mathbb{Q} -factorial:

Claim. *Let $f : X \rightarrow Y$ be a purely inseparable finite morphism between normal varieties. If X is \mathbb{Q} -factorial, so is Y .*

Proof. Everything follows from the following subclaim, which can be shown easily:

Subclaim. *Under the same assumptions, let $(U, f_U)_U$ be a Cartier divisor on X , where $X = \bigcup U$ is an open covering of X and $f_U \in k(X)$. Then the pushforward of the Weil divisor corresponding to $(U, f_U)_U$ corresponds to the Cartier divisor $(U, N(f_U))_U$, where $N = N_{X/Y} : k(X) \rightarrow k(Y)$ is the norm function.*

Let D be a Weil divisor on Y . By assumption, there exists a positive integer m such that $m f^* D = f^* m D$ is Cartier. By the subclaim above, we can show that $f_* f^* m D = m \deg(f) D$ and that $f_* f^* m D$ is Cartier. \square

10. AMPLIFICATIONS

In this section, we extend our results

- to the case when varieties involved are not necessarily \mathbb{Q} -factorial.
- to Mori dream regions.

As an example of a Mori dream region, we treat the Shokurov polytopes in the final subsection.

10.1. Non- \mathbb{Q} -factorial MDS. In [AHL, §2], the notion of MDS has been extended to not necessarily \mathbb{Q} -factorial normal projective varieties. In this subsection we call them non- \mathbb{Q} -factorial MDS, and show that our main results are also valid in that context.

Definition 10.1. Let X be a normal projective variety. X is said to be a non- \mathbb{Q} -factorial MDS if

- (1) $\text{Cl}(X)$ is finitely generated, where $\text{Cl}(X)$ denotes the Weil divisor class group of X .
- (2) A Cox ring of X (see below for the definition) is of finite type over the base field.

Definition 10.2. Let X be a normal projective variety with finitely generated $\text{Cl}(X)$. Let $\Gamma \subset \text{WDiv}(X)$ be a finitely generated subgroup of rank $\dim_{\mathbb{Q}} \text{Cl}(X)_{\mathbb{Q}}$ which maps injectively to $\text{Cl}(X)_{\mathbb{Q}}$. A Cox ring of X is the multi-section ring

$$R_X(\Gamma) = \bigoplus_{D \in \Gamma} H^0(X, \mathcal{O}_X(D))$$

for such Γ .

Note that our definitions of non- \mathbb{Q} -factorial MDS and Cox rings coincide with those of bq -factorial MDSs when the variety is \mathbb{Q} -factorial. Note also that the finite generation of a Cox ring is independent of the choice of Γ .

Remark 10.3. Our definition of a Cox ring is different from that in [AHL, §2], but they differ up to taking the quotient by a finite group. The properties we are interested in, like finite generation, are invariant under such operations, so there is no significant difference between them.

In the proof of [AHL, Lemma 2.4], they used the characteristic zero assumption so that they can apply the existence of resolutions. Actually we can avoid the use of resolutions as follows, so that their results work in arbitrary characteristics.

Proof. We give a proof of [AHL, Lemma 2.4] which works in arbitrary characteristics.

Take prime divisors D_1, \dots, D_r on X which generates $\text{Cl}(X)$. Let $X' \rightarrow X$ be a birational proper morphism from a projective variety X' , whose existence is guaranteed by the Chow's lemma. Let D'_i be the strict transforms D_i 's. Now let $X'' \rightarrow X'$ be (the normalization of) the successive blow-ups of X' along D'_i 's. Let D''_i be the total transform of D'_i . Note that D''_i is a Cartier divisor by the construction of blow-up. Let $f : X'' \rightarrow X$ be the composition of the two morphisms, and note that $f_* D''_i = D_i$ holds.

If we regard f as the birational morphism π in the proof of [AHL, Lemma 2.4], the rest of the arguments works similarly. \square

We go back to our results. First, Theorem 1.1 also holds in this case:

Theorem 10.4. *Let X be a non- \mathbb{Q} -factorial MDS, and $X \rightarrow Y$ be a surjective morphism to another normal projective variety. Then Y also is a non- \mathbb{Q} -factorial MDS.*

Proof. The proof is essentially the same as that for Theorem 1.1, so we only point out where should be modified in the original one. First of all, we can replace X with its small \mathbb{Q} -factorization: i.e. there exists a small birational morphism $\tilde{X} \rightarrow X$ from a \mathbb{Q} -factorial normal projective variety \tilde{X} . This fact implicitly appears in [AHL, Theorem 2.3] and its proof.

We should check that $\dim_{\mathbb{Q}} \text{Cl}(Y)_{\mathbb{Q}} < \infty$. Note that $\text{Cl}(Y)$ does not change if we remove the singular locus of Y . If we remove the inverse image of this locus from X , the Weil divisor class group does not increase. The rest of the argument is the same.

In order to prove the finite generation of a Cox ring of Y , we take the Stein factorization of $X \rightarrow Y$ as before.

Nothing has to be changed for finite morphisms. For algebraic fiber spaces, we again remove the singular locus of Y and its inverse image. Then we can apply Lemma 3.2, since any Weil divisor on a non-singular variety is Cartier so that it can be pulled-back. \square

For a non- \mathbb{Q} -factorial MDS X , we can define the notion of Mori equivalence, Mori chambers and so on exactly in the same manner as before. On the other hand, we can take a small \mathbb{Q} -factorization $\tilde{X} \rightarrow X$ and the divisor class group and Mori equivalence do not change under this operation. Therefore we obtain

Theorem 10.5. *Let $X \rightarrow Y$ be a surjective morphism between non- \mathbb{Q} -factorial MDSs. Then*

$$\text{Fan}(Y) = \text{Fan}(X)|_{\text{Cl}(Y)_{\mathbb{R}}}$$

holds.

Proof. By taking suitable small \mathbb{Q} -factorizations of X and Y , the morphism lifts to the one between ordinary MDSs. Thus we can reduce the problem to our original Theorem 1.2. \square

10.2. Mori dream regions. Let X be a normal \mathbb{Q} -factorial projective variety.

There is a notion called Mori dream regions (MDR, for short) defined in [HK, Definition 2.12], which generalizes MDSs. In this subsection we check that our main results can be extended to MDRs.

First we recall the definition of MDRs from [HK, Definition 2.12]:

Definition 10.6. Let X be a normal \mathbb{Q} -factorial projective variety. A cone $\mathcal{C} \subset \text{Pic}(X)_{\mathbb{R}}$ spanned by finitely many line bundles is called a Mori dream region (MDR) if the multi-section ring

$$R_X(\mathcal{C}) = \bigoplus_{D \in \mathcal{C}} H^0(X, \mathcal{O}_X(D))$$

is of finite type over the base field.

If the natural morphism $\text{Pic}(X)_{\mathbb{Q}} \rightarrow \text{N}^1(X)_{\mathbb{Q}}$ is isomorphic and $\mathcal{C} = \text{Eff}(X)$, \mathcal{C} is a MDR if and only if X is a MDS.

As in the case of MDS, MDR can be characterized via the existence of a decomposition into finitely many rational polyhedral subcones such that on each of them the Zariski decomposition is \mathbb{Q} -linear:

Proposition 10.7. *Let X be a normal \mathbb{Q} -factorial projective variety and $\mathcal{C} \subset \text{Pic}(X)_{\mathbb{R}}$ be a cone spanned by finitely many line bundles. It is a MDR if and only if the following conditions are satisfied:*

- $\mathcal{C} \cap \text{Eff}(X)$ is spanned by finitely many line bundles.
- The section ring of any line bundle of $\mathcal{C} \cap \text{Eff}(X)$ is of finite type over the base field. In particular, those line bundles admit the Zariski decompositions (see §2.3).
- There exists a decomposition of $\mathcal{C} \subset \text{Pic}(X)_{\mathbb{R}}$ into finitely many rational polyhedral subcones such that on each of them the Zariski decomposition is \mathbb{Q} -linear.

Proof. ‘if’ part is exactly the same as the proof of Lemma 3.2. For the ‘only if’ part, the second condition follows from [HK, ‘If’ part of Lemma 1.6]. For the first and the third conditions, see [CL, Theorem 3.5]. \square

Remark 10.8. In [HK, Theorem 2.13], they claim that we can find a decomposition of \mathcal{C} into chambers \mathcal{C}_i so that for each of them we can find a contracting birational map $g_i : X \dashrightarrow Y_i$ such that

$$\mathcal{C}_i = \mathcal{C} \cap (g_i^* \text{Nef}(Y_i) * \text{ex}(g_i))$$

holds.

The author believe that it is not so easy to prove, since using that claim we can derive the existence of a minimal model from that of the canonical model. This is why he replaced [HK, Theorem 2.13] with Proposition 10.7.

We can introduce the notion of Mori equivalence and the fan structure on MDRs exactly in the same way as before, and our main results also generalizes. For a normal \mathbb{Q} -factorial projective variety X and a MDR \mathcal{C} , we denote thus defined fan structure on \mathcal{C} by $\text{Fan}(X; \mathcal{C})$. Then

Corollary 10.9. *Let $f : X \rightarrow Y$ be a surjective morphism between normal \mathbb{Q} -factorial projective varieties. Let $\mathcal{C} \subset \text{Pic}(X)_{\mathbb{R}}$ be a finitely generated rational polyhedral cone which is a MDR. Then $\mathcal{C}|_{\text{Pic}(Y)_{\mathbb{R}}}$ also is a MDR.*

Corollary 10.10. *Let $f : X \rightarrow Y$ as above, and set $\mathcal{C}' = \mathcal{C}|_{\text{Pic}(Y)_{\mathbb{R}}}$. Then $\text{Fan}(X; \mathcal{C})|_{\text{Pic}(Y)_{\mathbb{R}}} = \text{Fan}(Y; \mathcal{C}')$ holds.*

They can be proved exactly in the same manners as in those for Theorem 1.1 and 1.2.

10.3. Application to Shokurov polytopes. We consider an application to the Shokurov polytopes.

Example 10.11. Suppose that $p = 0$, X be a \mathbb{Q} -factorial normal projective variety and Δ_i ($i = 1, 2, \dots, N$) be effective divisors on X such that (X, Δ_i) are klt. We consider the case when the cone \mathcal{C} spanned by the divisors $(K_X + \Delta_i)$ is a MDR. In [BCHM], it was proven that it is the case when the boundary divisors Δ_i are big. This is expected to be true without the bigness assumption, but it still is widely open.

Here we note that for this \mathcal{C} , the fan structure $\text{Fan}(X; \mathcal{C})$ is the same as the decomposition discussed in [Ka].

Suppose that a minimal model and the canonical model exists for any log canonical divisor in \mathcal{C} . Note that this implies that \mathcal{C} is a MDR. In this situation, subcone V_j of \mathcal{C} defined in [Ka, Theorem 3], which is a cone of log canonical divisors with the same log canonical models, is a Mori equivalence class. Moreover subcone $W_{j,k}$ of V_j in [Ka, Theorem 4], which is a cone of log canonical divisors sharing the same log canonical model and a minimal model, is a strong Mori equivalence class: i.e. the relative interior of a cone of $\text{Fan}(X; \mathcal{C})$.

Let $X \rightarrow Y$ be a surjective morphism. From our Corollary 10.9 and 10.10, the restriction \mathcal{C}' of \mathcal{C} to $\text{Pic}(Y)_{\mathbb{R}}$ again is a MDR and its fan structure comes from that of \mathcal{C} . Using the Stein factorization and the adjunction theorems [A, Theorem 4.1] and [FG, Lemma 1.1], we can easily see that \mathcal{C}' is also spanned by finitely many klt log canonical divisors. In view of this, our Corollary 10.10 applied to this case can be interpreted as the comparison theorem of log minimal (resp. log canonical) models of X and Y .

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GRADUATE SCHOOL OF MATHEMATICAL SCIENCES, THE UNIVERSITY OF TOKYO,
3-8-1 KOMABA, MEGURO-KU, TOKYO 153-8914, JAPAN.

E-mail address: okawa@ms.u-tokyo.ac.jp